§6.5 Consistency Conditions

Useful to assume that <u>all</u> symmetry currents (even global symmetries) are coupled to gauge fields. At the end, we can always take $g \to 0$ for these symmetries and return to a global symmetry

$\longrightarrow$ apart from anomalies, the effective action $\Gamma[A]$ is invariant under

background gauge field

$$A_{\beta\mu}(y) \longmapsto A_{\beta\mu}(y) + i\int d^4x\, \varepsilon_\alpha(x)\, \mathcal{T}_\alpha(x) A_{\beta\mu}(y)$$

where we must take

$$-i\,\mathcal{T}_\alpha(x) = -\frac{\partial}{\partial x^\mu}\frac{\delta}{\delta A_{\alpha\mu}(x)} - C_{\alpha\beta\gamma} A_{\beta\mu}(x)\frac{\delta}{\delta A_{\gamma\mu}(x)}$$

in order to reproduce

$$\delta A^\beta{}_\mu = \partial_\mu \varepsilon^\beta + i\varepsilon^\alpha \underbrace{(t^A{}_\alpha)^\beta{}_\gamma}_{=\,-i C^\beta{}_{\gamma\alpha}} A^\gamma{}_\mu$$

Taking anomalies into account, have

$$\mathcal{T}_\alpha(x)\Gamma[A] = G_\alpha[x;A]$$

where

recall

$$D_\mu \langle J^\mu_\alpha(x) \rangle = -i\, G_\alpha[x; A]\,, \quad \delta_{gauge} S = \int \lambda_\alpha D_\mu J^\mu_\alpha$$

gauge trf. parameter

and $$\langle J^\mu_\alpha(x) \rangle = \frac{\delta}{\delta A_{\alpha\mu}(x)} \Gamma[A]$$

with $D_\mu = \partial_\mu - i A^\beta_\mu(x) (t_\beta)_e{}^m$ the gauge-covariant derivative

The commutation relations

$$[\tilde{\delta}_\alpha(x), \tilde{\delta}_\beta(y)] = i\, C_{\alpha\beta\gamma}\, \delta^4(x-y)\, \tilde{\delta}_\gamma(x)$$

imply the "Wess-Zumino" consistency conditions:

$$\begin{aligned} &\tilde{\delta}_\alpha(x)\, G_\beta[y; A] - \tilde{\delta}_\beta(y)\, G_\alpha[x; A] \\ &= i\, C_{\alpha\beta\gamma}\, \delta^4(x-y)\, G_\gamma[y; A] \end{aligned} \qquad (1)$$

Reformulate these in terms of BRST-trfs.:

introduce ghost field ω_α

$\longrightarrow$ define nilpotent BRST operator s:

$$s A_{\alpha\mu} = \partial_\mu \omega_\alpha + C_{\alpha\beta\gamma} A_{\beta\mu} \omega_\gamma\,,$$

$$s \omega_\alpha = -\frac{1}{2} C_{\alpha\beta\gamma}\, \omega_\beta \omega_\gamma\,,$$

Work with functional

$$G[\omega, A] = \int \omega_\alpha(x)\, G_\alpha[x; A]\, d^4x = i\, \delta_{gauge} S$$

with gauge trf. parameter $\lambda_\alpha = \omega_\alpha$!

$$\rightarrow \; s\,G[\omega,A] = -\frac{1}{2} C_{\alpha\beta\gamma} \int d^4x\, \omega_\beta(x)\omega_\gamma(x)\, G_\alpha[x;A]$$

$$- \int d^4x\, \omega_\alpha(x) \int d^4y \left[\frac{\partial \omega_\beta(y)}{\partial y^\mu} + C_{\beta\gamma\delta} A_{\gamma\mu}(y)\omega_\delta(y) \right] \frac{\delta G_\alpha[x;A]}{\delta A_{\beta\mu}(y)}$$

$$= \int d^4x \int d^4y\, \omega_\alpha(x)\, \omega_\beta(y) \Big[-\frac{1}{2} C_{\alpha\beta\gamma}\, \delta^4(x-y)\, G_\gamma[x;A]$$

$$+ i\, \mathcal{T}_\beta(y)\, G_\alpha[x;A] \Big]$$

As ghost fields anti-commute, write

$$s\,G[\omega,A] = -\frac{1}{2} i \int d^4x \int d^4y\, \omega_\alpha(x)\, \omega_\beta(y)$$

$$\times \Big[i C_{\alpha\beta\gamma}\, \delta^4(x-y)\, G_\gamma[x;A] + \mathcal{T}_\beta(y)\, G_\alpha[x;A] - \mathcal{T}_\alpha(x)\, G_\beta[y;A] \Big]$$

$\longrightarrow$ (1) holds iff

(2) $\quad s\,G[\omega,A] = 0 \quad$ for all ghost fields $\omega_\alpha(x)$

Now consider possibility that

$$G[\omega;A] = s\,F[A]$$

As s satisfies $s^2 = 0$, we automatically get $s\,G = 0$. Since $s F[A]$ is $\delta_{gauge} F[A]$ with $\lambda_\alpha = \omega_\alpha$ and $G[\omega;A]$ is the anomalous trf. of the action S, the modified action

$$S - s\,F[A] \qquad (*)$$

will be anonomaly free!

$\rightarrow$ we are interested in anomalies which cannot be subtracted as in (*)

$\rightarrow$ these are in the "cohomology" of the s-operator: $Ker(s)/Im(s)$

Write the anomaly as $G = \int d^4x\, \mathcal{G}(x)$

Then $\quad sG = 0 \iff s\mathcal{G}(x) = \partial_\mu \mathcal{X}^\mu(x)$ (3)

$\mathcal{X}^\mu$: ghost number $= 1$

Also write $sF[A] = \int d^4x\, s\underbrace{\mathcal{F}(x)}_{\text{ghost number } 0}$

$\rightarrow$ cohomology of s at ghost number 1:

$H^1(s|d)$

Now recall triangle anomaly for 3 flavor currents:

$$\left[\langle \partial_\mu J^\mu_\alpha(x)\rangle_\Delta\right]_{anom} = -\frac{1}{24\pi^2} D_{\alpha\beta\gamma}\, \varepsilon^{\kappa\nu\lambda\rho}\partial_\kappa A^\beta_\nu(x)\partial_\lambda A^\gamma_\rho(x)$$

$\rightarrow$ of dimensionality 4 (in units of energy)

WZ-consistency condition (1) relates operators of same dimensionality

$$\rightarrow G_\alpha = i\left[\langle \partial_\mu J^\mu_\alpha\rangle\right]_{anom} \qquad (4)$$

$$= -\frac{i}{24\pi^2}\varepsilon^{\kappa\nu\lambda\rho}\, Tr\left\{T_\alpha\left[\partial_\kappa A_\nu \partial_\lambda A_\rho + i c_i \partial_\kappa A_\nu A_\lambda A_\rho\right.\right.$$

$$+ ic_2 A_\kappa \partial_\nu A_\lambda A_\rho + ic_3 A_\kappa A_\nu \partial_\lambda A_\rho - c_4 A_\kappa A_\nu A_\lambda A_\rho]\},$$

where $A_\mu \equiv A_{\alpha\mu} T_\alpha$, and c_i are constants to be determined.

$\longrightarrow$ rewrite in language of differential forms:

- $dx^\kappa \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho = \varepsilon^{\kappa\nu\lambda\rho} d^4x, \quad d^4x = dx^0 dx^1 dx^2 dx^3$
- $d \equiv dx^\mu \frac{\partial}{\partial x^\mu}, \quad d^2 = 0, \quad ds + sd = 0$
- anti-commuting quantities

$$A \equiv i A_\mu dx^\mu = i A_{\alpha\mu} T_\alpha dx^\mu, \qquad \omega \equiv i\omega_\alpha T_\alpha$$

$\longrightarrow$ eq. (4) becomes:

$$G[\omega, A] = \frac{1}{24\pi^2} \int \mathrm{Tr} \Big\{ \omega \big[(dA)^2 + c_1 (dA) A^2$$
$$+ c_2 A (dA) A + c_3 A^2 (dA) + c_4 A^4 \big] \Big\}$$

rewrite BRST trfs. as:

$$sA = -d\omega + \{A, \omega\}$$
$$s\omega = \omega^2$$

$$\rightarrow s\,\mathrm{Tr}[\omega A^4] = \mathrm{Tr}\Big[\omega^2 A^4 - \omega \cancel{\{A,\omega\}} A^3 + \omega A \cancel{\{A,\omega\}} A^2$$
$$- \omega A^2 \cancel{\{A,\omega\}} A + \omega A^3 \cancel{\{A,\omega\}}$$
$$+ \omega\, d\omega\, A^3 \text{ terms}$$

$= \mathrm{Tr}[w^2 A^4] + w\,dw A^3$ terms

Thus $s\,G[w, A] = 0$ can only be satisfied iff $c_4 = 0$.

$$\rightarrow s\,G = \frac{1}{24\pi^2}\int \mathrm{Tr}\Big\{ -(dA)^2 w^2 + w\,dw\,A\,dA - dw\,w\,dA\,A$$
$$- A\,w\,dA\,dw - w\,A\,dw\,dA$$
$$+ c_1\left[w\,dA\,dw\,A - w\,dA\,A\,dw\right]$$
$$+ c_2\left[w\,dw\,dA\,A - w\,A\,dA\,dw\right]$$
$$+ c_3\left[w\,dw\,A\,dA - w\,A\,dw\,dA\right]$$
$$- c_1\left[-w\,A\,dw\,A^2 + w\,dw\,A^3 + w\,dA\,A^2 w\right]$$
$$- c_2\left[w\,A^2 dw\,A - w\,A\,dw\,A^2 + w\,A\,dA\,dw\right]$$
$$- c_3\left[-w\,A^3 dw + w\,A^2 dw\,A + w\,A^2 dA\,w\right]\Big\}$$

$\rightarrow$ can write the integrand as $d\mathcal{F}$ if

$c_1 = -c_2 = +c_3 = -\frac{1}{2}$

In this case, we get

$$G[w, A] = \frac{1}{24\pi^2}\int \mathrm{Tr}\Big\{ w\,d\big[\underbrace{A\,dA - \tfrac{1}{2}A^3}_{\text{Chern-Simons form}}\big]\Big\}$$

$$= \frac{1}{24\pi^2}\int \mathrm{Tr}\Big\{ w\,d\big[A F + \tfrac{1}{2}A^3\big]\Big\}, \qquad (5)$$

where $F \equiv \frac{1}{2} i t_\alpha F_{\alpha\mu\nu}\, dx^\mu \wedge dx^\nu = dA - A^2 \qquad (*)$

Stora - Zumino descent equations:

take spacetime dimensionality to be $2n$

Note that

$$dF \overset{(*)}{=} -d(A^2) = -(dA)A + A(dA) = [A,F]$$

$\rightarrow$ $\mathrm{Tr}\, F^{n+1}$ is closed:

$$d\,\mathrm{Tr}\, F^{n+1} = (n+1)\mathrm{Tr}\{(dF)F^n\} = \mathrm{Tr}\{[A,F^{n+1}]\} = 0$$

$\rightarrow$ if spacetime is simply connected, $\mathrm{Tr}\{F^{n+1}\}$ is exact:

$$\mathrm{Tr}\{F^{n+1}\} = d\Omega_{2n+1}$$

Since $\mathrm{Tr}\{F^{n+1}\}$ is gauge invariant, and BRST trf. is a special gauge trf., we get

$$s\,\mathrm{Tr}\{F^{n+1}\} = 0$$

Using $sd + ds = 0$, we get

$$d(s\Omega_{2n+1}) = -s\,\mathrm{Tr}\{F^{n+1}\} = 0$$

$$\rightarrow \quad s\Omega_{2n+1} = d\Omega^1_{2n}$$

Furthermore, $d(s\Omega^1_{2n}) = -s^2\Omega_{2n+1} = 0$

$\rightarrow$ $s\Omega^1_{2n} = d\Omega^2_{2n-1}$ and we have

$$s\int_{\text{spacetime}} \Omega^1_{2n} = 0$$

Thus $\int \Omega^1_{2n}$ is a candidate for $G[\omega, A]$ by integrating the two 1st order diff. eqs.

$$d\,\Omega_{2n+1} = \mathrm{Tr}\{F^{n+1}\} \quad \text{and} \quad d\Omega^1_{2n} = s\,\Omega_{2n+1}$$

$\rightarrow$ solutions are given by:

$$\Omega_{2n+1} = (n+1)\int_0^1 dt\, \mathrm{Tr}\{A F_t^n\},$$

$$(**)\quad \Omega^1_{2n} = -(n+1)\sum_{r=0}^{n-1}\int_0^1 dt\,(1-t)\,\mathrm{Tr}\{\omega d(F_t^r A F_t^{n-1-r})\},$$

where $F_t \equiv tF + (t-t^2)A^2$.

Evaluation of $(**)$ shows that (5) gives a result for $G[\omega, A]$ proportional to $\int\Omega^1_4$ in case of 4 spacetime dimensions.